# Scaling behaviour of the relaxation in quantum chains

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**Abstract.** We consider the nonequilibrium time evolution of the transverse magnetization in the critical Ising and $XX$ quantum chains. For some inhomogeneously magnetized initial states we derive analytically the transverse magnetization profiles and show that they evolve into scaling forms in the long-time limit. In particular it is seen that the Ising chain exhibits some similarities with the conserved dynamics $XX$ chain. That is, after a transient regime, the total residual magnetization in the transverse direction is also conserved in the Ising case. A class of general initial states is also considered.

**PACS.** 75.40.Gb Dynamic properties (dynamic susceptibility, spin waves, spin diffusion, dynamic scaling, etc.)
05.70.Ln Nonequilibrium and irreversible thermodynamics
05.30-d Quantum statistical mechanics

## 1 Introduction

Nonequilibrium properties of quantum systems have attracted a lot of interest since they have natural dynamics in contrast to classical ones and since classical effects are actually quantal. There are several ways to consider nonequilibrium quantum systems. One is to couple the quantum mechanical system to a heat bath which can be itself described quantum mechanically [1]. In this case, a part of the whole system is isolated and called 'the system' while the rest is supposed to describe a certain environnement with which the system interacts and dissipates through. Another route is to impose a current on the system and investigate the steady states [2]. Still an other possibility is simply to investigate the relaxation of an initial state, in which the system has been prepared, and evaluate expectation values of observables at later times. This was done recently [3] on the $XX$-quantum chain with a step-like magnetization initial state. More recently in ref. [4], the relaxation of spatially inhomogeneous initial states has been treated for several variants of the XY quantum model. Relaxation phenomena at zero temperature with homogeneous initial state has been considered in ref. [5],[6] for the $XX$ and Ising chains in a transverse field in the context of aging.

In this work we study the nonequilibrium profiles of the critical Ising and $XX$ quantum chains. We suppose that at the initial time $t = 0$ the system is prepared in a given state $|\Psi\rangle$. The time evolution of the system is entirely governed by the Schrödinger equation and is formally given by

$$|\Psi(t)\rangle = \exp(-iHt)|\Psi\rangle \qquad (1)$$

since the systems under consideration are closed. The basic quantity we calculate is the expectation value of the local transverse magnetization at time $t$, $m(l,t) \equiv \langle\Psi(t)|\sigma_l^z|\Psi(t)\rangle$. We first consider two different initial states $|\Psi\rangle$, a kink in the $z$ direction, $|\ldots \uparrow\uparrow\uparrow\downarrow\downarrow\downarrow \ldots\rangle$, which was already considered in ref. [3] for the quantum $XX$ chain, and a droplet configuration $|\ldots \uparrow\uparrow\downarrow \ .. \ \downarrow\uparrow\uparrow \ldots\rangle$ for both Ising and $XX$ chains. In ref. [5] the cumulated magnetization was calculated for the $XX$ chain but the profile itself was not considered. We give finally the general expression for the relaxation of the transverse magnetization in terms of a convolution product in the continuum limit. The kernels of both XX and Ising chains are readily expressed in both direct and Fourier space.

## 2 Basic Quantities

The one dimensional Ising and $XX$ Hamiltonians with $L$ sites and open boundary conditions are given by the same one-parameter anisotropic $XY$ Hamiltonian:

$$H = -\frac{1}{2}\sum_{k=1}^{L-1}\left[\frac{1+\kappa}{2}\sigma_k^x\sigma_{k+1}^x + \frac{1-\kappa}{2}\sigma_k^y\sigma_{k+1}^y\right] - \frac{h}{2}\sum_{k=1}^{L}\sigma_k^z \qquad (2)$$

where the anisotropy parameter $\kappa = 1$ corresponds to the Ising case with a $Z_2$ symmetry and $\kappa = 0$ describes the $XX$-model which has $U(1)$ symmetry. The Hamiltonian (2) is diagonalizable through of a Jordan-Wigner transformation, followed by a canonical transformation [7]. In terms of the Clifford operators $\{\Gamma_l^i\}$, the

Jordan-Wigner transformation is expressed as [8]

$$\Gamma_n^1 = (-1)^{n-1} \left( \prod_{j=1}^{n-1} \sigma_j^z \right) \sigma_n^x$$
$$\Gamma_n^2 = -(-1)^{n-1} \left( \prod_{j=1}^{n-1} \sigma_j^z \right) \sigma_n^y \ . \tag{3}$$

The generators $\{\Gamma_l^i\}$ satisfy

$$\langle \Gamma_n^i | \Gamma_k^j \rangle = \delta_{ij} \delta_{nk} \ , \qquad (n, l = 1, \dots, L; \ i, j = 1, 2) \ , \tag{4}$$

where we have introduced a pseudoscalar product defined as $\langle C | D \rangle \equiv \frac{1}{2}\{C, D\}$ with $\{.,.\}$ the anticommutator. The original spin variables are obtained in terms of the $\Gamma$s by inverting the previous relations. One obtains $\sigma_n^x \sigma_{n+1}^x = -i\Gamma_n^2\Gamma_{n+1}^1$, $\sigma_n^y \sigma_{n+1}^y = -i\Gamma_{n+1}^2\Gamma_n^1$ and $\sigma_n^z = -i\Gamma_n^2\Gamma_n^1$, so that (2) is written as

$$H = \frac{i}{4} \sum_k -\Gamma_k^\dagger (i\sigma^y)\Gamma_{k+1} + \kappa \Gamma_k^\dagger \sigma^x \Gamma_{k+1} - h\Gamma_k^\dagger i\sigma^y \Gamma_k \ , \tag{5}$$

where $\Gamma_k^\dagger = (\Gamma_k^1, \Gamma_k^2)$, the hermitian conjugate of $\Gamma_k$, is a 2-components spinor and $\sigma^x$ and $\sigma^y$ are the Pauli matrices. Introducing the $2L$-components Clifford operator $\mathbf{\Gamma}^\dagger = (\Gamma_1^\dagger, \Gamma_2^\dagger, \dots, \Gamma_L^\dagger)$, we arrive at $H = (1/4)\mathbf{\Gamma}^\dagger \mathbf{T}\mathbf{\Gamma}$ with $\mathbf{T}^\dagger = \mathbf{T}$. The diagonalisation is then performed by the introduction of the diagonal Clifford generators $\gamma_q^\dagger = (\gamma_q^1, \gamma_q^2)$ related to the old one by $\Gamma_l^1 = \sum_q \phi_q(l)\gamma_q^1$ and $\Gamma_l^2 = \sum_q \psi_q(l)\gamma_q^2$ with real $\phi$ and $\psi$ components. Introducing the Fermi operators $\eta_q = 1/2(\gamma_q^1 + i\gamma_q^2)$ and the adjoint $\eta_q^\dagger$, finally one arrives at the usual free fermionic Hamiltonian $H = \sum_q \epsilon_q \eta_q^\dagger \eta_q + E_0$. The excitations energies $\epsilon_q$ and the transformation coefficients $\phi_q(l), \psi_q(l)$ are solution of the $2L \times 2L$ eigenvalue system $\mathbf{T}V_q = \epsilon_q V_q$, with components $V_q^\dagger(k) = (\phi_q(k), -i\psi_q(k))$. The eigenvectors satisfy the orthogonality relations $\sum_q \phi_q(i)\phi_q(j) = \delta_{ij}$ and $\sum_q \psi_q(i)\psi_q(j) = \delta_{ij}$.

The time evolution of the spin operators are easily expressed in terms of the time dependence of the Clifford generators $\Gamma$ (see Appendix). The basic time evolution of the diagonal operators is $\gamma_q(t) = e^{iHt}\gamma_q e^{-iHt} = \mathcal{R}(\epsilon_q t)\gamma_q$ where $\mathcal{R}(\theta)$ is a rotation of angle $\theta$. In matrix form we have

$$\begin{pmatrix} \gamma_q^1(t) \\ \gamma_q^2(t) \end{pmatrix} = \begin{pmatrix} \cos\epsilon_q t & \sin\epsilon_q t \\ -\sin\epsilon_q t & \cos\epsilon_q t \end{pmatrix} \begin{pmatrix} \gamma_q^1 \\ \gamma_q^2 \end{pmatrix} \ . \tag{6}$$

Using this relations, we can express the time dependence of the $\Gamma$s through an expansion onto the basis $\{\Gamma_k^i\}$:

$$\Gamma_n^j(t) = e^{iHt}\Gamma_n^j e^{-iHt} = \sum_{k,i} \langle \Gamma_k^i | \Gamma_n^j(t) \rangle \ \Gamma_k^i \tag{7}$$

with components

$$\langle \Gamma_k^1 | \Gamma_l^1(t) \rangle = \sum_q \phi_q(k)\phi_q(l)\cos\epsilon_q t$$

$$\langle \Gamma_k^1 | \Gamma_l^2(t) \rangle = \langle \Gamma_l^2 | \Gamma_k^1(-t) \rangle = -\sum_q \phi_q(k)\psi_q(l)\sin\epsilon_q t$$

$$\langle \Gamma_k^2 | \Gamma_l^2(t) \rangle = \sum_q \psi_q(k)\psi_q(l)\cos\epsilon_q t \ . \tag{8}$$

For the Ising chain, at the critical point $h = 1$, the basic contractions are obtain in a closed form. For open boundary conditions, the excitation energies $\epsilon_q = 2\sin(q/2)$ and eigenvectors $\phi$ and $\psi$ are [9]:

$$\phi_q(l) = (-1)^l \frac{2}{\sqrt{2L+1}} \cos q(l - 1/2)$$

$$\psi_q(l) = (-1)^{l+1} \frac{2}{\sqrt{2L+1}} \sin ql \tag{9}$$

with $q = (2p+1)\pi/(2L+1)$. In the thermodynamic limit $L \to \infty$, the contractions are then expressed in terms of Bessel functions $J_n(z)$ of integer order as [11]:

$$\langle \Gamma_k^1 | \Gamma_l^1(t) \rangle = \langle \Gamma_k^2 | \Gamma_l^2(t) \rangle = (-1)^{k+l} J_{2(l-k)}(2t)$$

$$\langle \Gamma_k^1 | \Gamma_l^2(t) \rangle = -(-1)^{k+l+1} J_{2(l-k)+1}(2t) \ . \tag{10}$$

For the $XX$-chain in a similar way one obtains

$$\langle \Gamma_k^1 | \Gamma_l^1(t) \rangle = (i)^{l-k} J_{l-k}(t) \begin{cases} \cos(ht) \ ; \ l - k = 2p \\ -i\sin(ht) \ ; \ l - k = 2p + 1 \end{cases}$$

$$\langle \Gamma_k^1 | \Gamma_l^2(t) \rangle = (i)^{l-k} J_{l-k}(t) \begin{cases} -\sin(ht) \ ; \ l - k = 2p \\ -i\cos(ht) \ ; \ l - k = 2p + 1 \end{cases} \tag{11}$$

and $\langle \Gamma_k^2 | \Gamma_l^2(t) \rangle = \langle \Gamma_k^1 | \Gamma_l^1(t) \rangle$.

In what follows we are interested in the time relaxation of the transverse magnetization. In the Heisenberg picture, we have, using the expansion (7), $\sigma_l^z(t) = -i\Gamma_l^2(t)\Gamma_l^1(t) = -i\sum_{k1,k2}^{i1,i2} \langle \Gamma_{k1}^{i1} | \Gamma_l^2(t) \rangle \langle \Gamma_{k2}^{i2} | \Gamma_l^1(t) \rangle \Gamma_{k1}^{i1} \Gamma_{k2}^{i2}$. The expectation value in the z-direction $|\Psi\rangle$ state is then

$$m(l, t) = \sum_k \left[ \langle \Gamma_k^2 | \Gamma_l^2(t) \rangle \langle \Gamma_k^1 | \Gamma_l^1(t) \rangle \right.$$
$$\left. - \langle \Gamma_k^1 | \Gamma_l^2(t) \rangle \langle \Gamma_k^2 | \Gamma_l^1(t) \rangle \right] \langle \Psi | \sigma_k^z | \Psi \rangle \tag{12}$$

since only the terms $\langle \Psi | \Gamma_k^2 \Gamma_k^1 | \Psi \rangle$ are non-vanishing in the state $|\Psi\rangle$.

## 3 Kink-like initial state

We consider first the initial state with a kink located at the origin

$$|\Psi\rangle = | \uparrow \rangle^{\otimes N/2} \otimes | \downarrow \rangle^{\otimes N/2} \tag{13}$$

where $| \uparrow, \downarrow \rangle$ are the eigenstates of the $\sigma^z$ Pauli matrix, $\sigma^z | \uparrow, \downarrow \rangle = \pm | \uparrow, \downarrow \rangle$, and we take the thermodynamic limit $N \to \infty$. We present here only the results for the critical ($h = 1$) Ising quantum chain since the $XX$ chain was already considered in ref. [3]. Using the previous formulas (12) and (10) for the contractions, a straightforward calculation leads to

$$m(l, t) = -\sum_{p=1-l}^{l-1} \left[ \left( 1 - \left( \frac{p}{t} \right)^2 \right) J_{2p}^2(2t) + J_{2p}^{'2}(2t) \right] \ . \tag{14}$$

Now the analysis proceeds along the same lines as in ref. [3]. We introduce the discrete derivative

$$\Phi_n^{'}(v) \equiv -t[m(n+1,t) - m(n,t)]_{n/t=v}$$
$$= -2\frac{n}{v}[(1-v^2)J_{2n}^2(2t) + J_{2n}^{'2}(2t)] \quad (15)$$

For obvious symmetry reasons, we will consider only the part $v > 0$ since we have $m(-n,t) = -m(n,t)$. Due to the different asymptotic properties of the Bessel functions one has to distinguish between the cases $v > 1$ and $v < 1$. For $v > 1$, which means that $n > t$, we are in the acausal region (outside the light-cone) since the excitations, traveling with velocity one [10], have no time to propagate from the initial position of the kink to the site $n$. Then it is the local environnement which completly governs the behavior of the magnetization. That is, the magnetization relaxes as if the initial state was the completely ordered state and one has

$$m(n,t) = -\frac{1}{2} - \frac{1}{4t}J_1(4t) \quad (16)$$

for $n > t$. The local magnetization reaches the constant value $-1/2$ with corrections of order $t^{-3/2}$ so that the derivative $\Phi^{'}(v)$ is essentially vanishing for large $n$. This is exactly what is seen from the asymptotic behavior of the Bessel functions and their derivatives, vanishing as $\exp[-\lambda(v)n]$ with $\lambda(v) > 0$ [11].

Inside the light-cone ($v < 1$), with the help of the asymptotics for $\nu \gg 1$ [11]

$$J_\nu(\frac{\nu}{\cos\beta}) = \sqrt{\frac{2}{\pi\nu\tan\beta}}\cos\psi$$
$$J_\nu^{'}(\frac{\nu}{\cos\beta}) = -\sqrt{\frac{\sin 2\beta}{\pi\nu}}\sin\psi \quad (17)$$

where $\psi \equiv \nu(\tan\beta - \beta) - \pi/4$, one obtains for the derivative $\Phi_n^{'}(v)$:

$$\Phi_n^{'}(v) = -\frac{2}{\pi}\sqrt{1-v^2} = \Phi^{'}(v) \quad (18)$$

which is $n$ independent due to the exact cancellation of the sin and cos terms in (15). Finally, by simple integration we obtain $m(n,t) = \Phi(n/t)$ with the scaling function

$$\Phi(v) = \begin{cases} 1/2 & v < -1 \\ -\frac{1}{\pi}[v\sqrt{1-v^2} + \arcsin v] & -1 < v < 1 \\ -1/2 & v > 1 \end{cases} \quad (19)$$

This has to be compared with the $XX$ chain result [3] $-\frac{2}{\pi}\arcsin v$ and $\pm 1$ outside the causal region. Contrary to the $XX$ chain which has a conserved dynamics (the total z-component of the magnetization is a constant of motion) in the Ising quantum chain there is a transient regime where the local magnetization relaxes faster than $t^{-1}$ toward the stationary value $\pm 1/2$ and then only the residual kink spreads as in the $XX$ chain. The results are shown in figure 1 where the inset describes the initial transient regime.
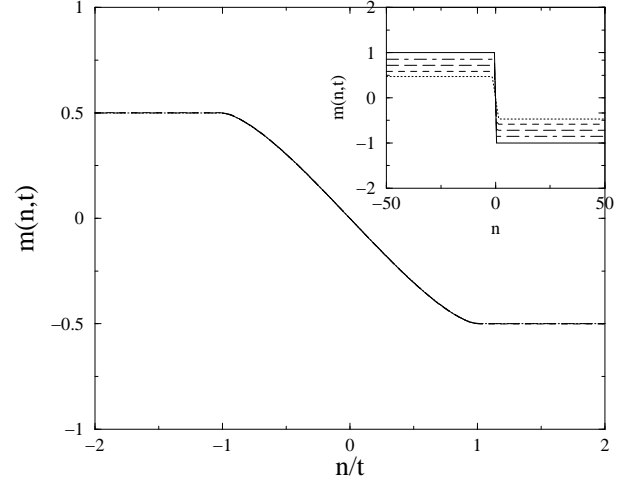


**Fig. 1.** Nonequilibrium transverse magnetization scaling function for the Ising quantum chain. The analytical expression (19) and the numerical results are indistinguishable. In the inset, the transient regime is shown for times smaller than $t = 2$. The magnetization relaxes toward the value $\pm 1/2$.

## 4 Droplet-like initial state

Let us consider now the following initial state

$$|\Psi\rangle = |\dots\uparrow\uparrow\uparrow\Downarrow\uparrow\uparrow\uparrow\dots\rangle \quad (20)$$

with $|\Downarrow\rangle = |\downarrow\rangle^{\otimes L}$, that is a droplet of $L$ down spins inside a bath of up spins, both interacting and evolving with the quantum Hamiltonian $H$. This can be considered as a toy model for a quantum system (the midle part) coupled to some environnement (the external part), both governed by the same microscopic interactions and one can study how the system part relaxes due to the coupling to the external degrees of freedom. We start with a one spin droplet within the Ising model. In this case, using (12) together with (20), the Ising transverse magnetization is given by

$$m(l,t) = \frac{1}{2} + \frac{1}{4t}J_1(4t) + \frac{1}{t}\Phi_l^{'}(l/t) \quad (21)$$

where $\Phi_l^{'}(v)$ is the function introduced in the previous section. For $v > 1$ the magnetization is dominated by the first two terms since then the function $\Phi_l^{'}(l/t)$ is exponentially small. On the other hand for $v < 1$, after the faster relaxation toward $1/2$ a scaling regime emerges for the local excess magnetization, that is

$$m^c(l,t) \equiv m(l,t) - 1/2 = t^{-1}\Phi^{'}(\frac{l}{t}) \quad (22)$$

where $\Phi^{'}(v)$ is given by (18).

More generaly, for a droplet of size $L$, the transverse magnetization at time $t$ is given by

$$m(l,t) = \frac{1}{2} + \frac{1}{4t}J_1(4t) + \frac{1}{t}\sum_{k=-L/2}^{L/2}\Phi_{l-k}^{'}((l-k)/t) . \quad (23)$$

In the light-cone the excess magnetization is given to the dominant order in $t^{-1}$ by

$$m^c(l,t) = t^{-1} \int_{-L/2}^{L/2} \Phi'((l-k)/t)\mathrm{d}k = \int_{l/t-L/2t}^{l/t+L/2t} \Phi'(u)\mathrm{d}u \tag{24}$$

so that finally in the scaling regime $l \gg L$ we have simply

$$m^c(l,t) = \frac{L}{t}\Phi'\left(\frac{l}{t}\right) = -\frac{2L}{\pi t}\sqrt{1-\left(\frac{l}{t}\right)^2} . \tag{25}$$

For $l \sim \mathcal{O}(L)$, with $J_n(z) \sim \sqrt{2/\pi z}\cos(z-n\pi/4-\pi/4)$ for large $z$, the local magnetization excess is simply $-2L/\pi t$ plus subdominant corrections. Then the total magnetization remaining inside the initial droplet region is

$$M^c(t) = \int_L m^c(l,t)\mathrm{d}l = -\frac{2L^2}{\pi t} \tag{26}$$

which is exactly what was obtained in ref. [5] for the $XX$ quantum chain. As in the $XX$ case, where the total magnetization is conserved, something similar happens in the Ising case. If one considers the total magnetization excess at time $t$, given by the integral over the whole space, we simply have a constant:

$$\int_{-\infty}^{\infty} m^c(l,t)\mathrm{d}l \simeq L \int_{-1}^{1} \Phi'(v)\mathrm{d}v = -L \tag{27}$$

where $-L$ is the residual excess magnetization after the initial transient regime since the up(down) domain relaxes locally toward $1/2(-1/2)$. This means that after the initial loss of magnetization, which takes place on microscopic time scales of order $t \simeq 1/h = 1$, the dynamic is conservative. We have a conservative deviation around the stationary value. In fact, one can show that the local magnetization excess satisfy a lattice continuity equation $\partial_t m^c(l,t) + j(l,t) - j(l-1,t) = 0$, with the current density $j(l,t)$ given by

$$j(l,t) = -2 \sum_{k=-L/2}^{L/2} J_{2(l-k)-1}(2t)J_{2(l-k)}(2t)$$
$$-J_{2(l-k)-2}(2t)J_{2(l-k)+1}(2t) \tag{28}$$

which is related to the expectation value of $\Gamma_{l-1}^2\Gamma_l^2 \propto \sigma_{l-1}^x\sigma_l^y$. In the continuum limit, the current is simply expressed as $j(x,t) = \frac{x}{t}m^c(x,t)$.

Although, the total residual magnetization in the system part was calculated in ref. [5] for the $XX$ chain, the scaling profile was not considered. Using equations (11) and (12) we have for all values of the transverse field $h$

$$m^c(l,t) \equiv m(l,t) - 1 = -2 \sum_{k=-L/2}^{L/2} J_{l-k}^2(t) . \tag{29}$$

In the light-cone, together with the asymptotic expressions for the Bessel functions, we obtain in the scaling regime $t > l \gg L$

$$m^c(l,t) = \frac{L}{t}\Psi\left(\frac{l}{t}\right) \tag{30}$$

with the scaling function

$$\Psi(v) = -\frac{2}{\pi}\frac{1}{\sqrt{1-v^2}} . \tag{31}$$

One may verify that the integral over the whole space of the local magnetization gives back $-2L$ as it should be for the conserved dynamics system under consideration. The scaling functions for both Ising and $XX$ chains are presented in figure 2, where the analytical results are compared with numerics.
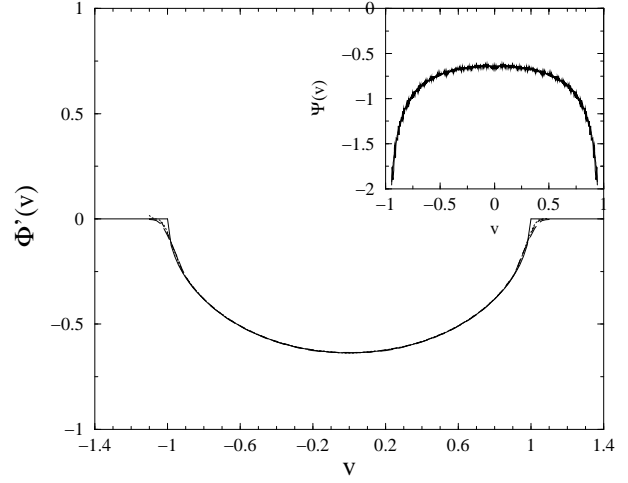


**Fig. 2.** Scaling functions for the critical Ising and $XX$(inset) quantum chains. The oscillations in the $XX$ case are finite size effects due to the initial droplet size. One can see that as the time is increased the numerical results are closer and closer to the analytical scaling function. This can be seen more evidently at the boundaries $v = \pm 1$ for the Ising chain.

# 5 General initial $z$-state

Clearly, for translation invariant Hamiltonians, equation (12) giving the transverse magnetization is a discret convolution product:

$$m(l,t) = \sum_{k=-\infty}^{\infty} F_t(l-k)S(k) = (F_t * S)(l) \tag{32}$$

with $S(k) = \langle\Psi|\sigma_k^z|\Psi\rangle$. The kernel $F_t(l)$ is given in the continuum limit by $F_t(l) = \frac{1}{t}f(\frac{l}{t})$ with

$$f_\kappa(v) = \begin{cases} \frac{1}{\pi}\left(1-v^2\right)^{\kappa-1/2} & |v| < 1 \\ 0 & |v| > 1 \end{cases} \tag{33}$$

where the $\kappa = 1$ refers to the Ising case and $\kappa = 0$ to the $XX$ chain. The local magnetization $m(x,t) = m_t(v)$, is then expressed as the convolution product

$$m_t(v) = (S_t * f)(v) \tag{34}$$

with $S_t(v) = S(tv)$. For the kink like initial state, we have $S_t(v) = -sgn(v) = 1 - 2H(v)$, where $H(v)$ is the Heaviside function. For the droplet-like initial state,

$$S_t(v) = 1 - \frac{2L}{t}\left(\frac{t}{L}\right)\Pi(\frac{tv}{L}) , \qquad (35)$$

where $\Pi(x)$ is the caracteristic function of the interval $[-1/2, 1/2]$. In the long time limit, $t \gg L$, we have $S_t(v) = 1 - \frac{2L}{t}\delta(v)$, so that we recover very simply the results of the previous section.

For a general initial state in the z-direction, the Fourier transform of equation (32) is

$$\tilde{m}_t(q) = \tilde{S}_t(q)\tilde{f}(q) \qquad (36)$$

with

$$\tilde{f}_\kappa(q) = \frac{1}{\pi}\int_{-1}^{1}(1-v^2)^{\kappa-1/2}e^{-2i\pi qv}dv \qquad (37)$$

so that the kernels in Fourier space are simply given by

$$\tilde{f}_0(q) = J_0(2\pi q) \qquad (38)$$

for the XX chain and

$$\tilde{f}_1(q) = \frac{J_1(2\pi q)}{2\pi q} = \frac{1}{2}\big(J_0(2\pi q) + J_2(2\pi q)\big) \qquad (39)$$

for the Ising chain. By inverse Fourier transform, it is possible to obtain the desired magnetization profile in direct space. For exemple, if we consider the modulated initial state $S(x) = \cos(2\pi x/L)$, with $L \gg 1$, we obtain in the long time regime $t \gg L$

$$m(x,t) = \cos(2\pi x/L)f_\kappa(2\pi t/L) , \qquad (40)$$

that is the modulation does not spread with time but only the amplitude is decreasing as $t^{-1/2-\kappa}$.

Finally, for a homogeneous initial state with $S(x) = m(0)$, it is easy to see that $m(t) = m(0)$ in the XX case, since the dynamic is conservative, and $m(t) = 1/2m(0)$ for the Ising chain.

## 6 Summary

We have calculated for the critical Ising and $XX$ quantum chains the nonequilibrium transverse magnetization profiles for kink-like and droplet-like initial states. In both cases at large times the magnetization profiles exhibit scaling forms $t^{-1}F(l/t)$ which have been obtained analytically and are in excellent agreement with the numerics. The two systems show essentially the same features even if the dynamics of the transverse magnetization are very different, conservative for the $XX$ chain and nonconservative for the Ising model. In the Ising case there is a transient regime where the initial magnetization relaxes toward the homogeneously initialy magnetized state stationary value $\pm 1/2$. After this initial regime, the system evolves as if the dynamics of the residual transverse magnetization was conservative. The long time relaxation of the transverse magnetization starting with a general initial $z$-state is expressed very simply in terms of a convolution product of the initial distribution with a response kernel $f_\kappa$ obtained analytically for both XX and Ising chain.

## Appendix: Time evolution

The diagonalisation of the Hamiltonian (5) leads to [8]

$$H = i\sum_q \frac{\epsilon_q}{2}\gamma_q^1\gamma_q^2 . \qquad (41)$$

In the Fermi operator representation, with $\eta_q = 1/2(\gamma_q^1 + i\gamma_q^2)$ and $\eta_q^\dagger$ the hermitian conjugate, one obtains $H = \sum_q \epsilon_q \eta_q^\dagger \eta_q - (1/2)\sum_q \epsilon_q$. The time evolution of the Clifford operators is given by $U_q^\dagger(t)\gamma_q U_q(t)$ with

$$U_q(t) = \exp\left(\frac{\epsilon_q t}{2}\gamma_q^1\gamma_q^2\right) = \cos\frac{\epsilon_q t}{2} + \gamma_q^1\gamma_q^2\sin\frac{\epsilon_q t}{2} , \quad (42)$$

which leads to equation (6). Since $\{\gamma_q^i, \gamma_{q'}^j\} = 2\delta_{ij}\delta_{qq'}$, we can write equivalently for equation (6) $\gamma_q^i(t) = \sum_{j=1}^{2}\langle\gamma_q^j|\gamma_q^i(t)\rangle\gamma_q^j$, where the symbol $\langle .|.\rangle$ means the half of the anticommutator.

The time evolution of the $\Gamma$s is then expressed as

$$\Gamma_k^1(t) = \sum_q \phi_q(k)\cos(\epsilon_q t)\gamma_q^1 + \phi_q(k)\sin(\epsilon_q t)\gamma_q^2 \qquad (43)$$

and

$$\Gamma_k^2(t) = \sum_q -\psi_q(k)\sin(\epsilon_q t)\gamma_q^1 + \psi_q(k)\cos(\epsilon_q t)\gamma_q^2 \quad (44)$$

with initial values $\Gamma_k^1(0) = \sum_q \phi_q(k)\gamma_q^1$ and $\Gamma_k^2(0) = \sum_q \psi_q(k)\gamma_q^2$. Reinjecting in this expressions the inverse transforms $\gamma_q^1 = \sum_k \phi_q(k)\Gamma_k^1$ and $\gamma_q^2 = \sum_k \psi_q(k)\Gamma_k^2$ one finally arrives at equations (7) with components (8).

Formally, since the anticommutators $\{\Gamma_k^i, \Gamma_l^j\} = 2\delta_{ij}\delta_{kj}$ are all proportional to the identity operator, the set $\{\Gamma_k^i\}$ forms an orthonormal basis of a $2L$-dimensional linear vector space $\mathcal{E}$ with inner product defined by $\langle .|.\rangle \equiv \frac{1}{2}\{.,.\}$. Hence, every vector $X \in \mathcal{E}$ has a unique expansion $X = \sum_{i,k}\langle\Gamma_k^i|X\rangle\Gamma_k^i$. The string expression $X_1X_2...X_n$, with $X_j \in \mathcal{E}$, is a direct product vector of the space $\mathcal{E}_1 \otimes \mathcal{E}_2 \otimes ... \otimes \mathcal{E}_n$ which decomposition is

$$X_1X_2...X_n = \sum_{i_1,k_1,...,i_n,k_n}\langle\Gamma_{k_1}^{i_1}|X_1\rangle...\langle\Gamma_{k_n}^{i_n}|X_n\rangle\Gamma_{k_1}^{i_1}...\Gamma_{k_n}^{i_n}$$

$$(45)$$

Using this formalism, the local magnetization at time $t$ is given by

$$\sigma_l^z(t) = -i\Gamma_l^2(t)\Gamma_l^1(t)$$
$$= -i\sum_{i_1,k_1,i_2,k_2}\langle\Gamma_{k_1}^{i_1}|\Gamma_l^2(t)\rangle\langle\Gamma_{k_2}^{i_2}|\Gamma_l^1(t)\rangle\Gamma_{k_1}^{i_1}\Gamma_{k_2}^{i_2} \quad (46)$$

which is our starting point. One has then to consider the simple time-independant expectation values $\langle\Psi|\Gamma_{k_1}^{i_1}\Gamma_{k_2}^{i_2}|\Psi\rangle$, which are easiely obtained in the spin basis using the Jordan-Wigner expressions (3).

# References

1. U. Weiss, *Quantum dissipative systems* (2nd edition) Series in Modern Condensed Matter Physics-Vol. 10, 1999 (World scientific Singapore)
2. T. Antal, Z. Rácz and L. Sasvári, Phys. Rev. Lett. **78**, 167 (1997)
3. T. Antal, Z. Rácz, A. Rákos and G. M. Schütz, Phys. Rev. E **59**, 4912 (1999)
4. G. O. Berim, S. Berim and G. G. Cabrera arXiv:cond-mat/0105594
5. G. M. Schütz and S. Trimper, Europhys. Lett. **47**, 164 (1999)
6. F. Iglói and H. Rieger, Phys. Rev. Lett. **85**, 3233 (2000)
7. E. Lieb, T. Schultz and D. Mattis, Ann. Phys. (N.Y.) **16**, 407 (1961)
8. H. Hinrichsen, J. Phys. A: Math. Gen. **27** 5393 (1994)
9. L. Turban and B. Berche, J. Phys. A: Math. Gen. **26** 3131 (1993)
10. The normalisation of the Hamiltonian is such that the sound velocity is one.
11. I. S. Gradshteyn and I. M. Ryszhik, *Table of Integrals Series and Products* (Academic, London, 1980)